# FORM FINDING ANALYSIS FOR MEMBRANE STRUCTURES USING FORCE DENSITY METHOD

by

Arata Yoshida
Kajima Corporation, Information Processing Center
Tokyo, Japan

Massimo Majowiecki
Professor, Istituto di Tecnica delle Costruzioni, Facoltà di Ingegneria
Bologna, Italy

Haruji Tsubota
Kajima Corporation, Institute of Construction Technology
Tokyo, Japan

## SUMMARY

The force density method was developed for cable networks. In this paper, we apply this force density method[1] to find the initial equilibrium configuration of membrane structures approximately by using the relationship between membrane stress in triangular element and equivalent imaginary forces. Some examples of this analysis are illustrated.

## 1. INTRODUCTION

Cable reinforced membrane structures possess significant advantages over other conventional structures for enclosing large volume. They have many features and need special analysis different from those for conventional structures. The features are as follows.

(1) They are essentially the unstable structures, and the initial stress or the inner pressure must be applied to make them be stable.
(2) They do not have flexural stiffness.
(3) They are able to resist only tensile stress.

Considering above features, two phases of the analysis are necessary, i.e. one is the form finding analysis and the other is the structural analysis including large deformation and wrinkling effect.

Among solution methods for the form finding analysis for membrane structures, the Newton-Raphson method is widely used. But this method has some drawbacks as follows.

(1) Not only coordinates of boundary nodes but also assumed initial coordinates of free nodal points are necessary. It means that it takes much time to make input data.

(2) According to the assumed initial coordinate, converged solutions can not be always obtained, i.e. the calculation sometimes diverges.

So more practical and convenient method is necessary to solve the form finding problem. On the contrary the force density method, which was developed for cable networks, has advantages as follows.

(1) Because it is a linear analysis, the equilibrium configuration can be always obtained.

(2) Only boundary conditions and element data are necessary to solve the problem. Assumed coordinates of free nodal points are not necessary, so input data can be made more speedy.

Firstly, we describe how to apply the force density method when the cable stress or length is specified by designer. Then we apply this method to solve the form finding analysis for membrane structures.

At the beginning of the design phase of membrane structures, equilibrium configurations under various boundary configurations and stress distributions should be solved to check their feasibility. So this method can be a more practical and powerful strategy for designing membrane structures.

## 2. FORCE DENSITY METHOD

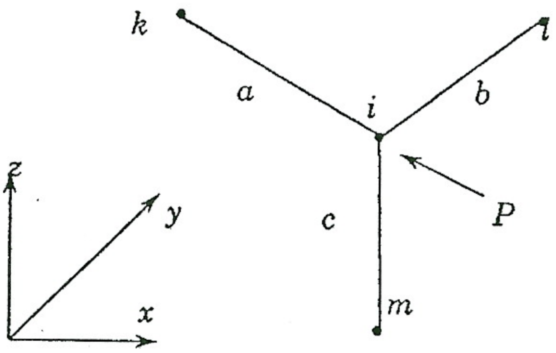



Figure 2.1 Cable elements converging at node $i$

In any point of a network consisting of pin-pointed bars (see Figure 2.1), the following equations of equilibrium of forces are derived[2],

$$\sum_j \frac{(x_i - x_j)}{L_{ij}} S_{ij} = P_{xi}$$

$$\sum_j \frac{(y_i - y_j)}{L_{ij}} S_{ij} = P_{yi} \qquad \cdots\cdots\cdots\cdots (2.1)$$

$$\sum_j \frac{(z_i - z_j)}{L_{ij}} S_{ij} = P_{zi}$$

where $x_i, y_i, z_i$ : the coordinate of node $i$,
$S_{ij}$ : the axial forces of the cable element $ij$,
$L_{ij}$ : the length of the cable element $ij$,
$\Sigma$ : summation of cable elements converging at node $i$,
$P_{xi}$ : external force acting node $i$.

By making equation (2.1) for every nodal point of the structure, a system of equations can be obtained.
Now we introduce new parameter in equation (2.1),

$$q_{ij} = S_{ij} / L_{ij} \qquad \cdots\cdots\cdots\cdots (2.2)$$

which is called the force density. Then equation (2.1) can be written as follows.

$$\begin{aligned} \sum_j (x_i - x_j) q_{ij} &= P_{xi} \\ \sum_j (y_i - y_j) q_{ij} &= P_{yi} \\ \sum_j (z_i - z_j) q_{ij} &= P_{zi} \end{aligned} \qquad \cdots\cdots\cdots\cdots (2.3)$$

If the designer specifies the force density $q$, equation (2.3) become linear equations with respect to coordinate $x$, $y$, $z$, and we can obtain following equation from equations (2.3).

$$x_i = \frac{\sum_j q_{ij} \cdot x_j + P_{xi}}{\sum_j q_{ij}} \qquad \cdots\cdots\cdots\cdots (2.4)$$

where $q_{ij}$ : the force density of the cable element $ij$
$\Sigma$ : summation of cable elements converging at node $i$

Equations about $y$ and $z$ coordinate have same form.
Iterating equation (2.4) for all free nodal component until change between the actual value and the former one becomes small enough, we can get the equilibrium configuration. Convergence condition can be written as follows.

$$| x_i^{n-1} - x_i^n | \simeq 0.0 \qquad \cdots\cdots\cdots\cdots (2.5)$$

where $n$ represent the number of iteration step. After calculating length of cable element using coordinates of this equilibrium configuration, axial force can be obtained as follows.

$$S = q \cdot L \qquad \cdots\cdots\cdots\cdots (2.6)$$

It is more naturally for the designer to specify axial force ( $S$ ) or length ( $L$ ) of cable rather than force density ( $q$ ). It is possible if the force density method is used iteratively until errors between specified value and calculated value become small enough as see in Figure (2.2),

where $S_0$ : specified axial force,
$L_0$ : specified cable length,
$e$ : tolerance factor,

and without suffix means actual value.

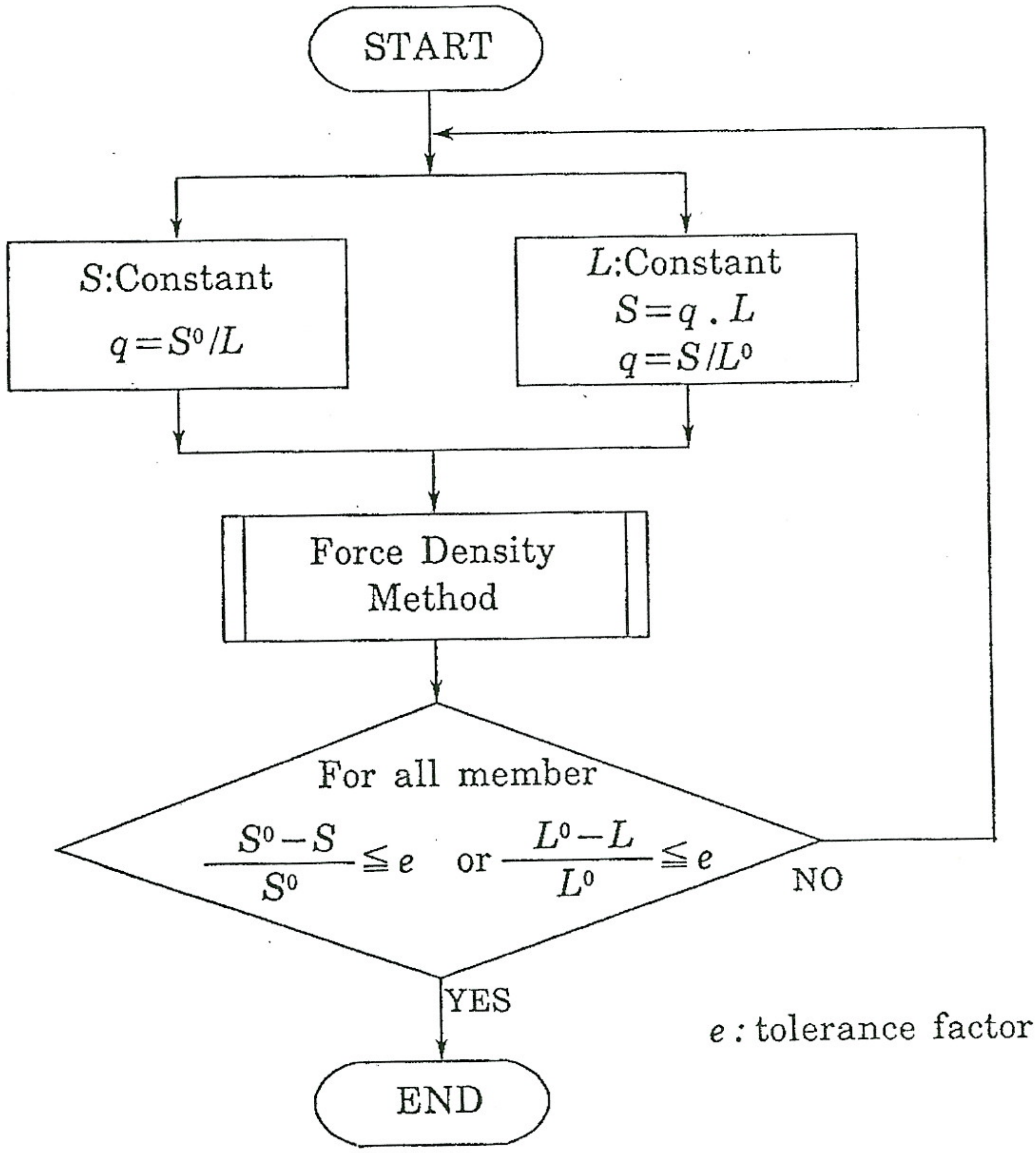



Figure 2.2

## 3. APPLICATION TO MEMBRANE STRUCTURES

In this chapter we will discuss how to apply the force density method to the form finding analysis for membrane structures. At first the triangular membrane element is decomposed to three cable elements. Then the initial configuration is calculated assuming that each force density ' q ' of all cable elements is equal to 1.0.

If it is assumed that the state of stress in triangular element is constant, we can get a linear relation between membrane stresses and equivalent imaginary forces as follows ,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} T \end{bmatrix} \cdot \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \cdots\cdots\cdots\cdots (3.1)$$

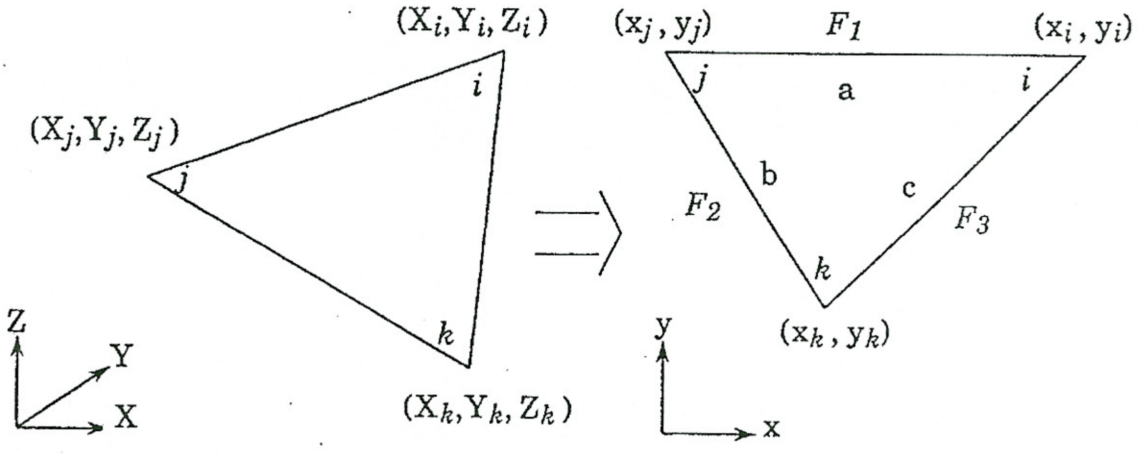
where $\sigma_x, \sigma_y, \tau_{xy}$ : membrane stress,
$F_1, F_2, F_3$ : equivalent imaginary forces.

As a relation between the global coordinate system and local one is presented as Figure 3.1, the explicit representation of transformation matrix [ $T$ ] and its inverse can be written as follows,

$$\begin{bmatrix} T \end{bmatrix} = \frac{2}{t} \begin{bmatrix} a/2\Delta & (a^2+b^2-c^2)^2/8\Delta a^2 b & (a^2-b^2+c^2)^2/8\Delta a^2 c \\ 0 & 2\Delta/a^2 b & 2\Delta/a^2 c \\ 0 & -(a^2+b^2-c^2)^2/2a^2 b & (a^2-b^2+c^2)^2/2a^2 c \end{bmatrix} \qquad \cdots\cdots\cdots\cdots (3.2)$$

$$\begin{bmatrix} T \end{bmatrix}^{-1} = \frac{t}{2} \begin{bmatrix} 2\Delta/a & -(a^2+b^2-c^2)(a^2-b^2+c^2)/8\Delta a & (b^2-c^2)/a \\ 0 & b(a^2-b^2+c^2)/4\Delta & b \\ 0 & c(a^2+b^2-c^2)/4\Delta & c \end{bmatrix} \qquad \cdots\cdots\cdots\cdots (3.3)$$

where $t$ : thickness of the membrane,
$\Delta$ : area of the triangular element,
$a, b, c$ : length of the side $ij$ and so on.




Global Coordinate　　　Local Coordinate

Figure. 3.1 Coordinate system

Secondly equivalent imaginary forces of cable elements are calculated using equation (3.3). Then the force density method is applied iteratively as described in the former chapter assuming that each cable axial force is constant. Finally the membrane stresses are calculated using equation (3.2), with cable axial forces and new coordinates. This step is repeated until the difference between specified membrane stresses and calculated ones becomes sufficiently small.
These processes are summarized in Figure 3.2.

## 4. NUMERICAL EXAMPLES

### 4.1. Spherical Pneumatic Structure

To demonstrate the accuracy of proposed method, the form finding problem of a spherical shaped pneumatic structure is solved. Conditions of the analysis are as follows.

Inner pressure : 0.25 t/m$^2$
Membrane stress : 1.0 t/m

Only fixed boundary coordinates are specified and other free nodal coordinate are assumed to be zero at first. Calculated shape is shown in Figure 4.1 and 4.2.
From the relation between the inner pressure and the membrane stress, the radius (R) of the sphere becomes 8.0 m. Table 4.1 shows comparison between calculated and theoretical values. These calculated values are derived under a condition in which the tolerance factor ($e_2$ in Figure 3.2) is 1.0E-3. It is found that the agreement is good enough.
Figure 4.3 shows the change of the maximum movement ( equation 2.5 ) and maximum unbalanced force. Figure 4.4 shows the change of maximum errors between specified imaginary axial force and calculated one ( see Figure 2.2 ).
Figure 4.5 shows the change of maximum error of specified membrane stress and calculated one.

### 4.2. H.P. Type Tensile Structure Bordered By Cables

H.P.type tensile structure is solved. Four edges are bordered by cables. Membrane stress is assumed to be 1.0 t/m and cable axial force is assumed to be 15.0 t. Coordinates of four corner points are specified. At first coordinates of other points are assumed to be all zero. Figure 4.6 to Figure 4.8 shows calculated shape.

## 5. CONCLUSIONS

We have discussed how to apply the force density method to the form finding analysis for membrane structures. Numerical examples of a pneumatic structure and a tensile structure are presented. Convergence is good enough and errors between specified stress and calculated value can be controlled easily. It is found that this method will be a powerful strategy for designing membrane structures.

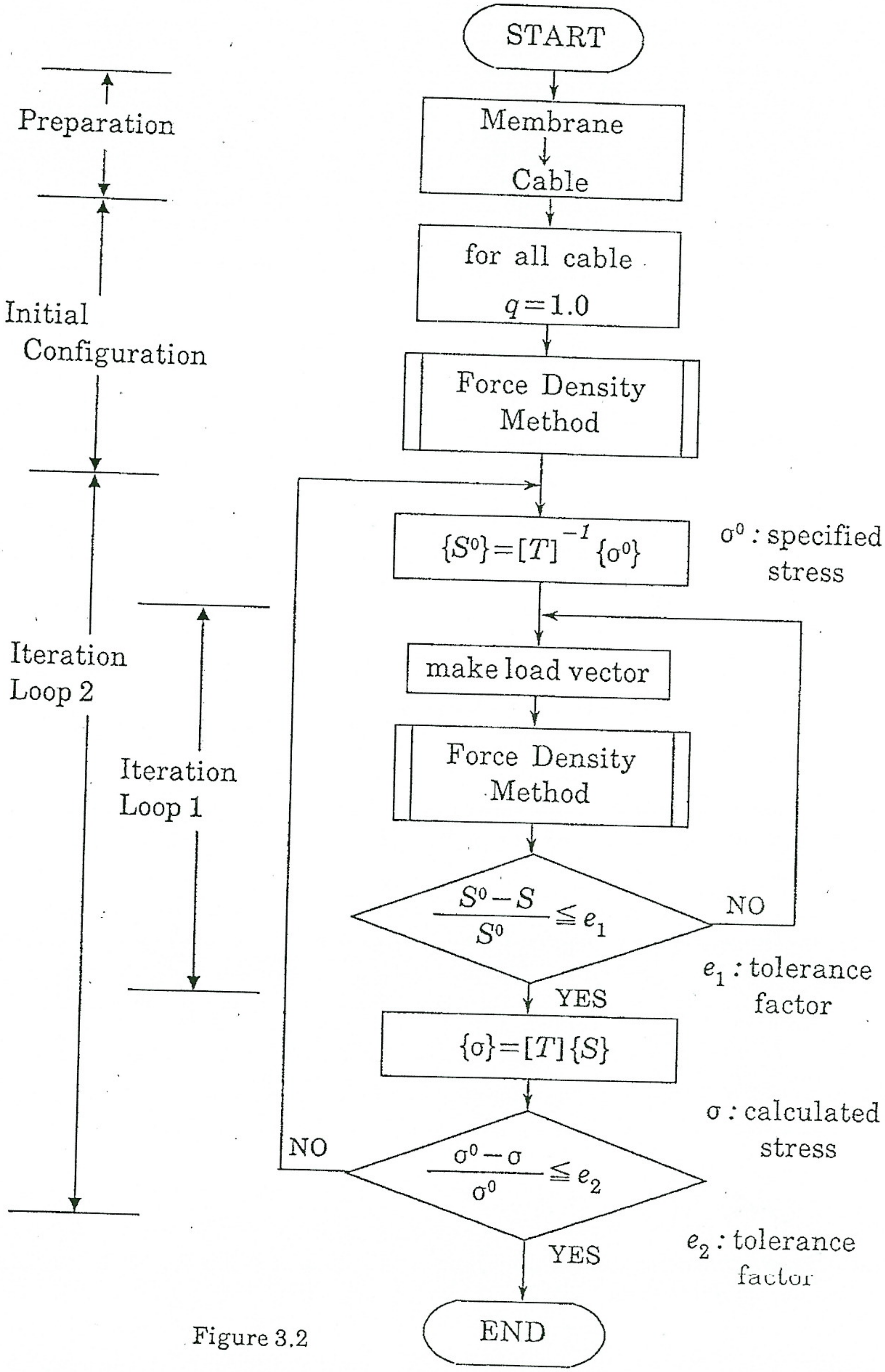

START
Membrane
Cable
for all cable
q=1.0
Force Density Method
{S⁰}=[T]⁻¹{σ⁰}
σ⁰ : specified stress
make load vector
Force Density Method
(S⁰−S)/S⁰ ≦ e1
NO
YES
e1 : tolerance factor
{σ}=[T]{S}
σ : calculated stress
(σ⁰−σ)/σ⁰ ≦ e2
NO
YES
e2 : tolerance factor
END
Preparation
Initial Configuration
Iteration Loop 2
Iteration Loop 1

Figure 3.2

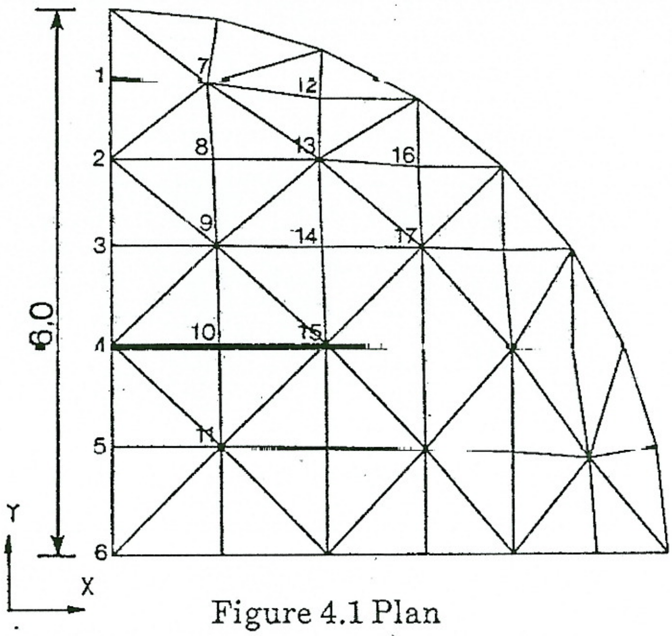



Figure 4.1 Plan

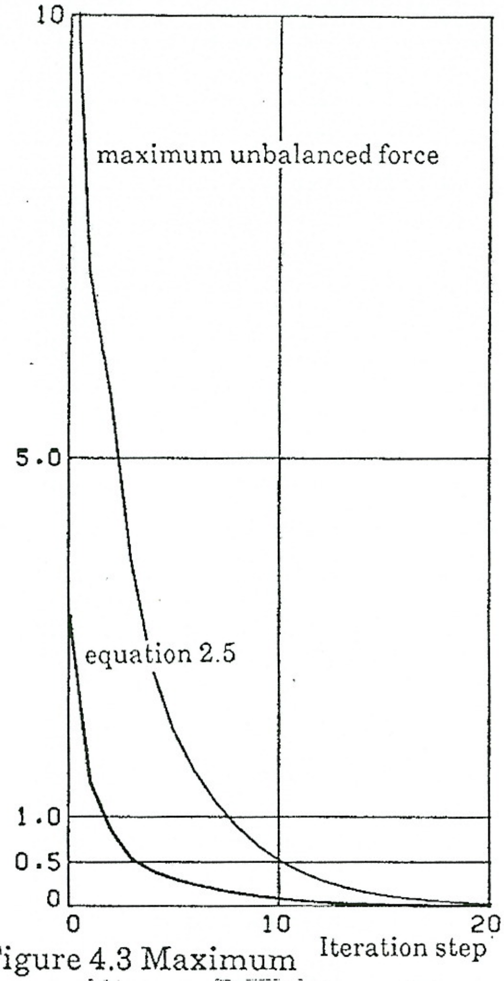



Figure 4.3 Maximum movement and unbalanced force

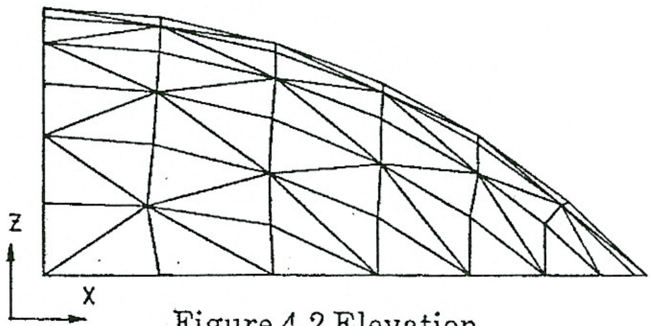



Figure 4.2 Elevation

| Node | X | Y | Z | R | Error(%) |
|---|---|---|---|---|---|
| 1 | 0.0 | 5.2444 | 0.7410 | 7.9935 | 0.0817 |
| 2 | 0.0 | 4.3419 | 1.4140 | 7.9885 | 0.1440 |
| 3 | 0.0 | 3.3954 | 1.9388 | 7.9878 | 0.1519 |
| 4 | 0.0 | 2.3223 | 2.3496 | 7.9861 | 0.1731 |
| 5 | 0.0 | 1.1871 | 2.6049 | 7.9851 | 0.1860 |
| 6 | 0.0 | 0.0 | 2.6931 | 7.9846 | 0.1923 |
| 7 | 1.1050 | 5.1709 | 0.7124 | 7.9930 | 0.0878 |
| 8 | 1.1127 | 4.3392 | 1.3259 | 7.9911 | 0.1116 |
| 9 | 1.1465 | 3.3902 | 1.8515 | 7.9894 | 0.1331 |
| 10 | 1.1725 | 2.3221 | 2.2604 | 7.9873 | 0.1582 |
| 11 | 1.1845 | 1.1855 | 2.5165 | 7.9858 | 0.1770 |
| 12 | 2.2666 | 4.9953 | 0.5247 | 7.9949 | 0.0640 |
| 13 | 2.2574 | 4.3329 | 1.0387 | 7.9963 | 0.0462 |
| 14 | 2.2879 | 3.3688 | 1.5848 | 7.9917 | 0.1040 |
| 15 | 2.3223 | 2.3212 | 1.9929 | 7.9902 | 0.1224 |
| 16 | 3.3331 | 4.2500 | 0.6020 | 7.9941 | 0.0743 |
| 17 | 3.3665 | 3.3648 | 1.1335 | 7.9960 | 0.0499 |

Table 4.1 Comparison between calculated and theoretical value

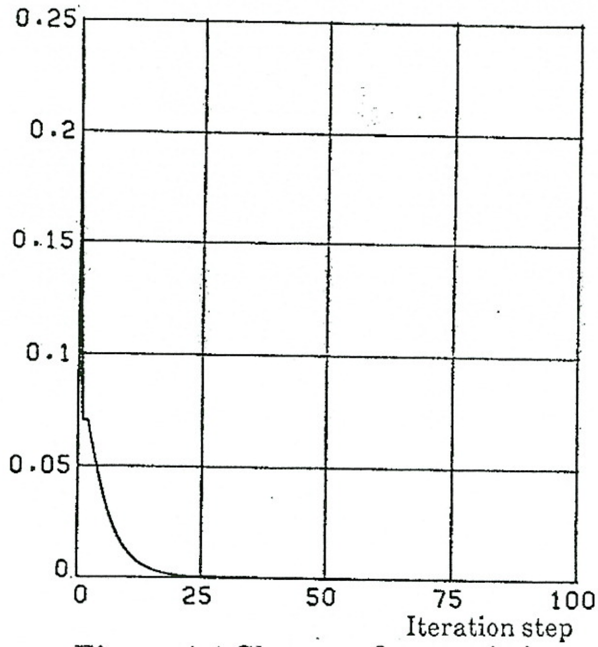



Figure 4.4 Change of errors ($e_1$)

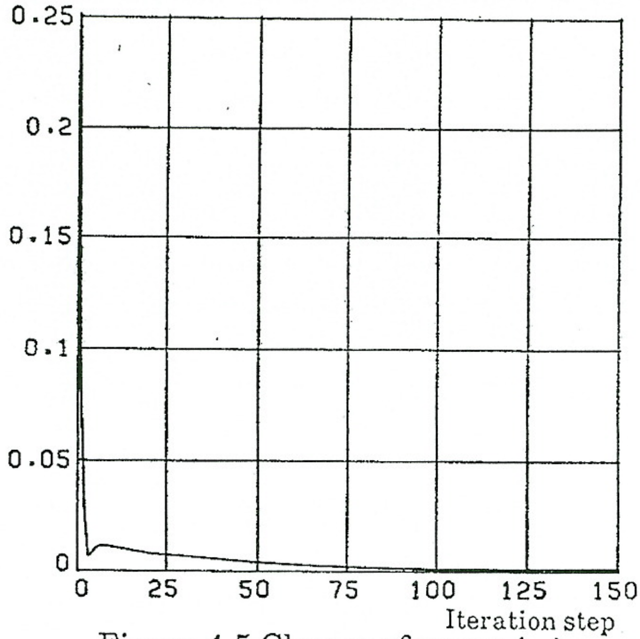



Figure 4.5 Change of errors ($e_2$)

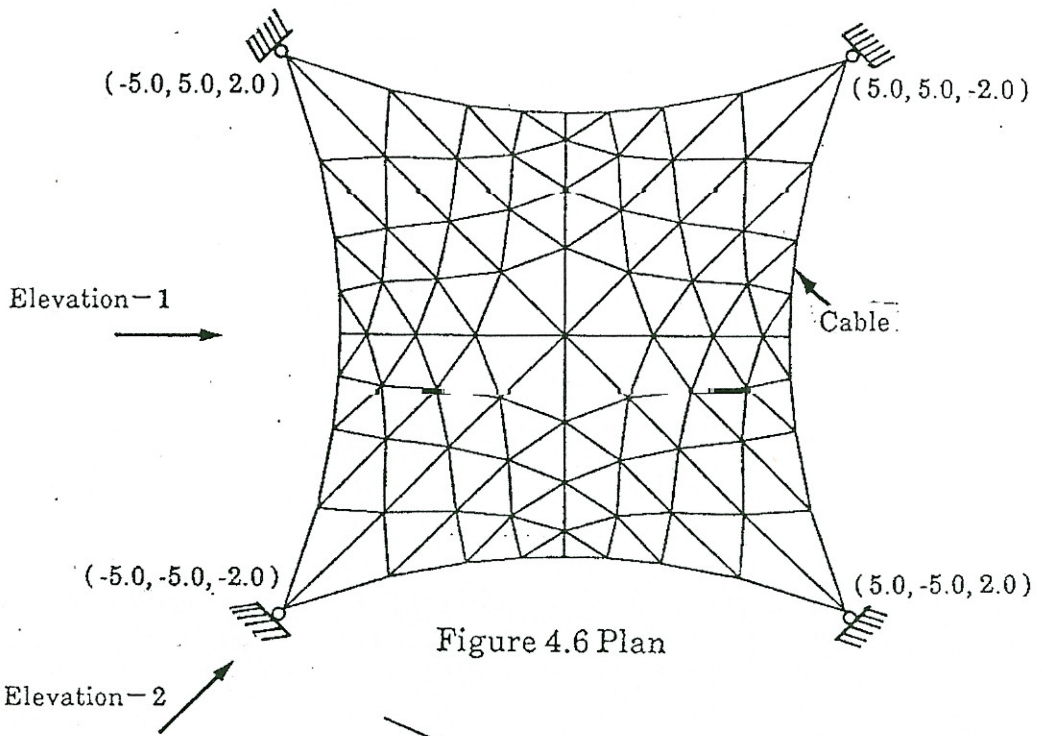



Figure 4.6 Plan

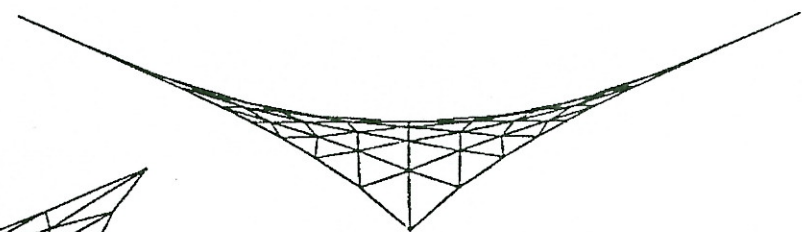


Figure 4.8 Elevation－2

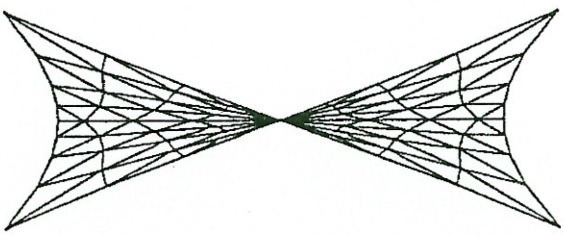


Figure 4.7 Elevation－1

## 6. REFERENCES


(1) H. J. Scheck
The Force Density Method for Form Finding and Computation of General Networks
Computer Methods in Applied Mechanics and Engineering, 1974

(2) L.Gründig and J. Bahndorf
Form Finding of a Roof Structure for a Health Spa
Light Structures in Architecture, Australia, 1986

(3) M. Majowiecki and D. H. Geiger
Large-span Pneumatic Structures Reinforced by Steel Cables
Acier Stahl Steel, 1977

(4) M. Majowiecki
Recent Developments of Design and Construction of Membrane Structures in Italy
I.A.S.S. Symposium, Osaka, 1986